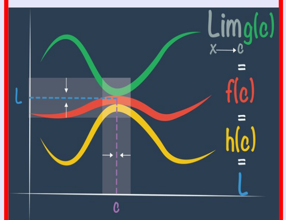
Math 261

Fall 2022

Lecture 22



Implicit Differentiation:

Find $\frac{dy}{dx}$ for $xy=4$.

Method I: Isolate y

$$y = \frac{4}{x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{4}{x}\right] = \frac{0\cdot x - 4\cdot 1}{x^2} = \frac{-4}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-4}{x^2}}$$

Method II: Implicit Differentiation

$$xy=4$$

Take derivative of both sides

$$\frac{d}{dx}[xy] = \frac{d}{dx}[4]$$

$$\overbrace{\frac{d}{dx}[x]}^{1}\cdot y + x\cdot\frac{d}{dx}[y] = 0$$

$$y + x\cdot\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \boxed{\frac{-y}{x}}$$

$$\frac{dy}{dx} = \frac{-\frac{4}{x}}{x} = \boxed{\frac{-4}{x^2}}$$

Find $\frac{dy}{dx}$ for $x^2 + y^2 = 25$

To isolate y $\rightarrow$ $y^2 = 25 - x^2$

$y = \pm\sqrt{25 - x^2}$

Find $\frac{dy}{dx}$

Using Implicit Diff.

$$x^2 + y^2 = 25$$

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[25\right]$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\rightarrow 2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Find $\frac{dy}{dx}$ for $x^4 - y^3 = 2xy$

$$\frac{d}{dx}\left[x^4 - y^3\right] = \frac{d}{dx}\left[2xy\right]$$

$$\frac{d}{dx}\left[x^4\right] - \frac{d}{dx}\left[y^3\right] = 2\frac{d}{dx}\left[xy\right]$$

$$4x^3 - 3y^2 \cdot \frac{dy}{dx} = 2\left(\cancel{\frac{d}{dx}\left[x\right]}^{1} \cdot y + x \cdot \frac{d}{dx}\left[y\right]\right)$$

$$4x^3 - 3y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

Now isolate $\frac{dy}{dx}$

$$4x^3 - 2y = 2x\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

$$4x^3 - 2y = \left(2x + 3y^2\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x^3 - 2y}{2x + 3y^2}$$

Find $\frac{dy}{dx}$ for $\cos y + x^2 = y - 8$

$$\frac{d}{dx}\left[\cos y + x^2\right] = \frac{d}{dx}\left[y - 8\right]$$

$$\frac{d}{dx}[\cos y] + \cancel{\frac{d}{dx}[x^2]}^{\,2x} = \frac{dy}{dx} - \cancel{\frac{d}{dx}[8]}^{\,0}$$

$$-\sin y \cdot \frac{dy}{dx} + 2x = \frac{dy}{dx}$$

$$2x = \frac{dy}{dx} + \sin y \cdot \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{1 + \sin y}}$$

Find $\frac{dy}{dx}$ for $\tan \frac{x}{y} = x + y$ $\quad \frac{d}{dx}\left[\frac{x}{y}\right]$

$$\frac{d}{dx}\left[\tan \frac{x}{y}\right] = \frac{d}{dx}\left[x + y\right]$$

$$\sec^2 \frac{x}{y} \cdot \frac{1 \cdot y - x \cdot \frac{dy}{dx}}{y^2} = 1 + \frac{dy}{dx}$$

Multiply by y^2

$$y^2 \sec^2 \frac{x}{y} \left(y - x \frac{dy}{dx} \right) = y^2 + y^2 \frac{dy}{dx}$$

Isolate $\frac{dy}{dx}$

$$y^3 \sec^2 \frac{x}{y} - x y^2 \sec^2 \frac{x}{y} \cdot \frac{dy}{dx} = y^2 + y^2 \frac{dy}{dx}$$

$$y^3 \sec^2 \frac{x}{y} - y^2 = \left(x y^2 \sec^2 \frac{x}{y} + y^2\right) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^3 \sec^2 \frac{x}{y} - y^2}{x y^2 \sec^2 \frac{x}{y} + y^2} = \frac{\cancel{y^2}\left(y \sec^2 \frac{x}{y} - 1\right)}{\cancel{y^2}\left(x \sec^2 \frac{x}{y} + 1\right)}$$

$y \neq 0$

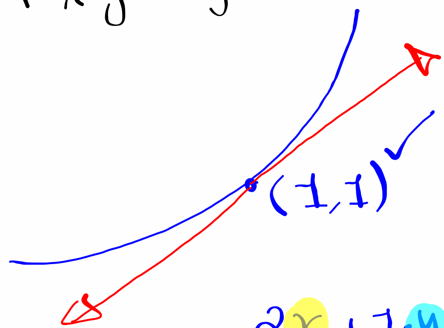
Double-check

Ans. from book

$$\frac{dy}{dx} = \frac{y \sec^2 \frac{x}{y} - y^2}{y^2 + x \sec^2 \frac{x}{y}}$$

Find eqn of tan. line to the curve

$x^2 + xy + y^2 = 3$ at $(1,1)$.



$m = \frac{dy}{dx}\Big|_{(1,1)}$

$(1,1)$ ✓

$$\frac{d}{dx}\left[x^2+xy+y^2\right] = \frac{d}{dx}[3]$$

$$2x + 1\cdot y + x\cdot\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Evaluate at $(1,1)$, and replace $\frac{dy}{dx}$ with m.

$$2(1) + 1\cdot 1 + 1\cdot m + 2\cdot 1\cdot m = 0$$

$$2 + 1 + m + 2m = 0$$

$$3m = -3 \longrightarrow \boxed{m=-1}$$

$$y - 1 = -1(x-1)$$

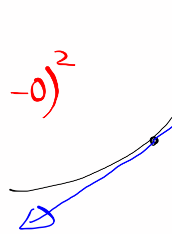
$$\boxed{y = -x + 2}$$

Find eqn of tan. line to the curve given by $x^2+y^2 = (2x^2+2y^2-x)^2$ at $(0,\frac{1}{2})$.

Verify the point

$$0^2 + \left(\tfrac{1}{2}\right)^2 = \left(2\cdot 0^2 + 2\cdot\left(\tfrac{1}{2}\right)^2 - 0\right)^2$$

$$\frac{1}{4} = \left(\tfrac{1}{2}\right)^2 \checkmark$$



$m = \frac{dy}{dx}\Big|_{(0,\frac{1}{2})}$

$(0,\frac{1}{2})$

$$x^2+y^2 = (2x^2+2y^2-x)^2$$

$$\overset{0}{\cancel{2x}} + 2y\cdot\frac{dy}{dx} = 2\overset{1}{(\overset{0}{\cancel{2x^2}}+2y^2-\overset{0}{\cancel{x}})}\cdot(\overset{0}{\cancel{4x}} + 4y\cdot\frac{dy}{dx} - 1)$$

$$2\left(\tfrac{1}{2}\right)\cdot m = \overset{1}{\cancel{2\left(2\cdot\left(\tfrac{1}{2}\right)^2\right)}}\cdot\left(4\cdot\tfrac{1}{2}\cdot m - 1\right)$$

$$m = 2m - 1$$

$$m - 2m = -1$$

$$-m = -1 \quad \boxed{m=1}$$

$$y - \frac{1}{2} = 1(x-0)$$

$$\boxed{y = x + \frac{1}{2}}$$

Given $x^2 + y^2 = r^2$ and $ax + by = 0$

1) Find $\frac{dy}{dx}$ for each equation. $\rightarrow ax = -by$

$x^2 + y^2 = r^2$

$2x + 2y\frac{dy}{dx} = 0$

$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$

$ax + by = 0$

$a + b\frac{dy}{dx} = 0$

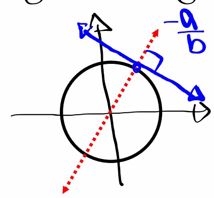
$\boxed{\frac{dy}{dx} = \frac{-a}{b}}$

2) Multiply these two $\frac{dy}{dx}$, and Simplify.

$$\frac{-x}{y} \cdot \frac{-a}{b} = \frac{ax}{by} = \frac{-by}{by} = -1$$

$x^2 + y^2 = r^2$ Circle

$ax + by = 0$ line



Orthogonal

Find $\frac{dy}{dx}$ for each eqn below:

Circle

$x^2 + y^2 = ax$

$2x + 2y\frac{dy}{dx} = a$

$\frac{dy}{dx} = \frac{a-2x}{2y}$

$x^2 + y^2 = by$

$2x + 2y \cdot \frac{dy}{dx} = b\frac{dy}{dx}$

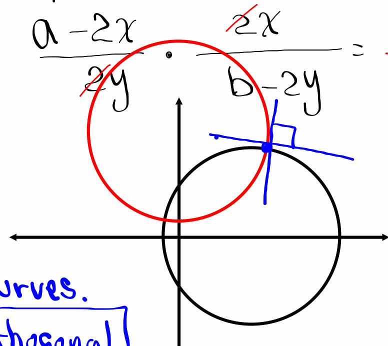
$\frac{dy}{dx} = \frac{2x}{b-2y}$

Multiply them, and Simplify

$$\frac{a-2x}{\cancel{2}y} \cdot \frac{\cancel{2}x}{b-2y} = \frac{ax-2x^2}{by-2y^2} = \frac{x^2+y^2-2x^2}{x^2+y^2-2y^2}$$

$$= \frac{y^2-x^2}{x^2-y^2} = \frac{-(x^2-y^2)}{x^2-y^2}$$

$$= -1$$



Curves.

$\boxed{\text{Orthogonal}}$